

Canonical Description of T-duality for Fundamental String and D1-Brane and Double Wick Rotation

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Abstract

We study T-duality transformations in canonical formalism for Nambu-Gotto action. Then we investigate the relation between world-sheet double Wick rotation and sequence of target space T-dualities and Wick rotation in case of fundamental string and D1-brane.

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1 Introduction and Summary

A double Wick rotation on the string world-sheet was proposed in [1] and now it is well known that it is a very useful tool for the study of AdS/CFT correspondence, for review see [2]. It is crucial that double Wick rotation has sense in case of the light cone gauge fixed string since only in this case the double Wick rotation transforms string world-sheet theory non-trivially. In fact, the thermodynamics of this different two dimensional quantum field theory gives the solution of the spectral problem of AdS/CFT when the spectrum of the string on $AdS_5 \times S^5$ is computed with the help of the thermodynamic Bethe ansatz for this mirror model [3, 4, 5, 6].

Due to the significance of the double Wick rotation one can ask the question whether this transformation has more physical meaning. Such a question was firstly posed in [9] and it was shown there that the double Wick rotation on the world-sheet of the gauge fixed bosonic string is equivalent to the particular transformation of the metric and NS-NS two form. This analysis was then extended to the case of the uniform gauge fixed Green-Schwarz superstring in [10] where it was shown that the double Wick rotation is equivalent to the particular transformation of metric, NS-NS two form and dilaton and Ramond-Ramond fields. It was also shown there that these transformations can be interpreted from a target space perspective as a combination of T dualities and analytic continuation, following A. Tsytlín's suggestion.

Due to the fact that the idea that the double Wick rotation could have deeper physical meaning is very attractive we applied it for another two dimensional theory which is a low energy effective action for D1-brane in [11]. In this paper we firstly determined uniform gauge fixed action for D1-brane in general background. Then we applied double Wick rotation for this theory and we found that the double Wick rotated action has the same form as the original one when the background gravitational and NS-NS two form fields transform as in [9] while we found that dilaton and Ramond-Ramond fields transform differently. We suggested that this discrepancy could be explained by the fact that D1-brane behaves differently under T-duality transformations but we leaved the detailed analysis of this issue for future. The aim of this paper is to answer this question. More precisely, we would like to see how suggested combinations of T-dualities and Wick rotation in the target space-time can be performed on the world-sheet of the fundamental string and D1-brane action. Due to the fact that the gauge fixing is important for the given procedure and since this gauge fixing is performed when we fix one of the components of momenta rather than target space coordinate we perform T-duality transformation on the level of the Hamiltonian formalism. In fact, the interpretation of T-duality transformation as canonical transformation was given in [12] ² in case of gauge fixed string action. We generalize this analysis to the case of the Nambu-Gotto (NG) action which is invariant under two dimensional diffeomorphism and we obtain celebrated Buscher's rules for the T-duality transformations of the background metric and NS-NS two form field [14, 15] ³.

²For review, see [13].

³We would like to stress that given procedure could be equivalently performed with Polyakov

Using this result we perform sequence of T-duality transformations and Wick rotation in the target space-time and we show that when we impose the uniform gauge in the resulting Hamiltonian we obtain that the Hamiltonian for the physical degrees of freedom is the same as in case of the double Wick rotated gauge fixed theory with an important exception that the sequence of T-duality transformation and Wick rotation do not generate a change $B_{\mu\nu} \rightarrow -B_{\mu\nu}$ where $B_{\mu\nu}$ are components of NS-NS two form fields that are transverse to the directions where corresponding T-duality and Wick rotations were performed. On the other hand it is well known that the transformation $B \rightarrow -B$ is the symmetry of supergravity equations of motion with the absence of RR fields. So that whenever string propagates on background with g, B that solves the supergravity equations of motion it can propagate on the background with $g, -B$. So that without loss of consistency we can augment the sequence T-duality and Wick rotation with an additional operation $B \rightarrow -B$. Then this extended sequence of operation is equivalent to the double Wick rotation of the uniform gauge fixed string theory⁴.

As the next step we extend this analysis to the case of D1-brane effective action which consists of Dirac-Born-Infeld (DBI) action and Wess-Zumino (WZ) term. We firstly apply the canonical transformations for this theory and we derive how the background fields transform. However we should stress that we are able to perform such an analysis when we presume that the electric flux is fixed. This is a natural requirement since it is well known that the electric flux is proportional to the number of the fundamental strings and we would like to compare two actions when the number of fundamental strings is the same. We find that under this canonical transformation the dilation and Ramond-Ramond fields transform differently than we should expect from T-duality rules which has very simple explanation. We argue that the final action corresponds to the fundamental string action moving in T-dual background when however the components of the background fields that appear in Buscher's rules correspond to the S-dual background when we use the equivalence of the D1-brane action (with constant electric flux) and fundamental string action in S-dual background [18].

Due to this fact we can now explain the discrepancy that we found in the previous paper [11]. Explicitly, we perform the sequence of canonical transformations and Wick rotation in case of Hamiltonian for D1-brane. Then we perform the uniform gauge fixing and we again find that the Hamiltonian for the physical degrees of freedom implies the transformation rules for the target space fields that have the same form as in case of the double Wick rotation performed on the uniform gauge fixed D1-brane effective action again with an exception that components of Ramond-Ramond two forms that are transverse to the directions of duality transformations do not transform. On the other hand we can again argue for the existence of the

form of the string action. Explicitly, the Hamiltonian has the same form as in case of NG string with subtle difference that now the primary constraints are the momenta conjugate to the components of the world-sheet metric. Then the requirement of their preservations gives the secondary constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$. Note that these constraints are the primary constraints of the NG string.

⁴The discussion is more subtle in case of non-zero RR fields and for detailed discussion see [10].

symmetry $C^{(2)} \rightarrow -C^{(2)}, B \rightarrow -B$ of the solutions of the supergravity equations of motion so that we find exact equivalence between double Wick rotation of the uniform gauge fixed D1-brane action and sequence of "canonical transformation-target space double Wick rotation-canonical transformation- $(C^2 \rightarrow -C^{(2)}, B \rightarrow -B)$ ".

Let us outline results derived in given paper. We generalize the canonical description of T-duality transformations for the string with no gauge fixing imposed. Then we show that the sequence of T dualities and Wick rotation in target space-time together with the uniform gauge fixing leads to the theory that is equivalent to the double Wick rotated gauge fixed theory. On the other hand we show that given procedure when applied to D1-brane action again gives theory that is equivalent to the double Wick rotated uniform gauge fixed theory which however cannot be interpreted as T-duality transformations.

This paper is organized as follows. In the next section (2) we determine T-duality transformations of the background fields as a canonical transformation of the Nambu-Gotto action for the fundamental string. In section (3) we perform sequence of T-duality transformations and Wick rotation in the Hamiltonian formulation of given theory and discuss its relation with the double Wick rotated uniform gauge fixed action. Section (4) is devoted to the generalization of given procedure to the case of D1-brane and finally in section (5) we perform the combinations of T-duality transformations and Wick rotation in case of Hamiltonian formulation of D1-brane theory.

2 T-duality for Fundamental String in Canonical Formalism

We would like to perform the analysis of T-duality as the canonical transformation of the NG action, following [12]. We start with the Nambu-Gotto action for the fundamental string

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \left[\sqrt{-\det g} + \frac{1}{2} \epsilon^{\alpha\beta} B_{MN} \partial_\alpha x^M \partial_\beta x^N \right], \quad (1)$$

where $g_{\alpha\beta} = G_{MN} \partial_\alpha x^M \partial_\beta x^N$, $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$ and where x^M , $M = 0, \dots, d$ label the embedding coordinates of the string. Further, $G_{MN}(x)$, $B_{MN}(x)$ are components of the background gravitational and NS-NS fields respectively.

As the first step we proceed to the Hamiltonian formulation of the action (1). From (1) we obtain the conjugate momenta

$$p_M = -\frac{1}{2\pi\alpha'} G_{MN} \partial_\alpha x^N g^{\alpha\tau} \sqrt{-\det g} - \frac{1}{2\pi\alpha'} B_{MN} \partial_\sigma x^N. \quad (2)$$

If we now define $\Pi_M = p_M + \frac{1}{2\pi\alpha'} B_{MN} \partial_\sigma x^N$ we obtain from (2) following primary

constraint

$$\mathcal{H}_\tau = (2\pi\alpha')\Pi_M G^{MN}\Pi_N + \frac{1}{(2\pi\alpha')}G_{MN}\partial_\sigma x^M\partial_\sigma x^N = 0 \quad (3)$$

together with the spatial diffeomorphism constraint

$$\mathcal{H}_\sigma = p_M\partial_\sigma x^M . \quad (4)$$

Then it can be shown that the bare Hamiltonian density defined as $H_B = p_M\partial_\tau x^M - \mathcal{L}$ vanishes identically and the extended Hamiltonian density is a sum of the primary constraints of the theory

$$\mathcal{H} = \lambda_\tau \mathcal{H}_\tau + \lambda_\sigma \mathcal{H}_\sigma , \quad (5)$$

where $\lambda_\tau, \lambda_\sigma$ are Lagrange multipliers corresponding to the constraints $\mathcal{H}_\tau \approx 0, \mathcal{H}_\sigma \approx 0$. It can be further shown that $\mathcal{H}_\tau, \mathcal{H}_\sigma$ are the first class constraints that are generators of two dimensional diffeomorphism of the world-sheet.

Let us presume that there is a direction in the target space-time that is invariant under constant shift

$$\theta \rightarrow \theta + \epsilon , \quad \epsilon = \text{const} . \quad (6)$$

In other words all background fields do not depend on θ . Our goal is to perform the canonical transformation from θ to $\tilde{\theta}$. Let us presume that this generating function has the form

$$G(\theta, \tilde{\theta}) = \frac{1}{4\pi\alpha'} \int d\sigma (\partial_\sigma \theta \tilde{\theta} - \theta \partial_\sigma \tilde{\theta}) , \quad (7)$$

where we presume that θ has canonical dimension $[\theta] = \text{length}$. Let us denote the momentum conjugate to $\tilde{\theta}$ as $p_{\tilde{\theta}}$. From the definition of the canonical transformations we derive following relations between canonical momenta p_θ and $p_{\tilde{\theta}}$

$$\begin{aligned} p_{\tilde{\theta}} &= -\frac{\delta G}{\delta \tilde{\theta}} = -\frac{1}{2\pi\alpha'} \partial_\sigma \theta , \\ p_\theta &= \frac{\delta G}{\delta \theta} = -\frac{1}{2\pi\alpha'} \partial_\sigma \tilde{\theta} . \end{aligned} \quad (8)$$

Now we obtain canonically dual Hamiltonian when we replace $\partial_\sigma \theta$ with $-(2\pi\alpha')p_{\tilde{\theta}}$ and p_θ with $-\frac{1}{2\pi\alpha'}\partial_\sigma \tilde{\theta}$ in \mathcal{H}_τ and \mathcal{H}_σ given in (3) and (4). Explicitly, we find

$$\begin{aligned} \tilde{\mathcal{H}}_\tau &= (2\pi\alpha')\hat{\Pi}_\mu G^{\mu\nu}\hat{\Pi}_\nu - 2(2\pi\alpha')\hat{\Pi}_\mu G^{\mu\nu}B_{\nu\theta}p_{\tilde{\theta}} + (2\pi\alpha')p_{\tilde{\theta}}B_{\mu\theta}G^{\mu\nu}B_{\nu\theta}p_{\tilde{\theta}} - \\ &- 2\hat{\Pi}_\mu G^{\mu\theta}\partial_\sigma \tilde{\theta} + 2\hat{\Pi}_\mu G^{\mu\theta}B_{\theta\nu}\partial_\sigma x^\nu + 2B_{\mu\theta}p_{\tilde{\theta}}G^{\mu\theta}\partial_\sigma \tilde{\theta} - 2B_{\mu\theta}p_\theta G^{\mu\theta}B_{\theta\nu}\partial_\sigma x^\nu + \\ &+ \frac{1}{2\pi\alpha'}G^{\theta\theta}(\partial_\sigma \tilde{\theta})^2 - 2\frac{1}{2\pi\alpha'}\partial_\sigma \tilde{\theta}G^{\theta\theta}B_{\theta\mu}\partial_\sigma x^\mu + \frac{1}{(2\pi\alpha')}B_{\theta\mu}\partial_\sigma x^\mu G^{\theta\theta}B_{\theta\nu}\partial_\sigma x^\nu + \\ &+ (2\pi\alpha')G_{\theta\theta}p_{\tilde{\theta}}^2 - 2G_{\mu\theta}\partial_\sigma x^\mu p_{\tilde{\theta}} + \frac{1}{(2\pi\alpha')}G_{\mu\nu}\partial_\sigma x^\mu\partial_\sigma x^\nu , \\ \tilde{\mathcal{H}}_\sigma &= p_\mu\partial_\sigma x^\mu + p_{\tilde{\theta}}\partial_\sigma \tilde{\theta} , \end{aligned} \quad (9)$$

where

$$\hat{\Pi}_\mu = p_\mu + \frac{1}{2\pi\alpha'} B_{\mu\nu} \partial_\sigma x^\nu, \quad (10)$$

$\mu, \nu = 0, 2, \dots, d$. We see from (9) that it is very difficult to identify the theory in dual picture. To do this it is more instructive to proceed to the Lagrangian formulation of the theory. Explicitly, with the help of (9) we derive following relations

$$\begin{aligned} \partial_\tau x^\mu &= \{x^\mu, H\} = 2\lambda_\tau [(2\pi\alpha') G^{\mu\nu} \hat{\Pi}_\nu - (2\pi\alpha') G^{\mu\nu} B_{\nu\theta} p_{\tilde{\theta}} - G^{\mu\theta} \partial_\sigma \tilde{\theta} + G^{\mu\theta} B_{\theta\nu} \partial_\sigma x^\nu] + \lambda_\sigma \partial_\sigma x^\mu, \\ \partial_\tau \tilde{\theta} &= \{\tilde{\theta}, H\} = 2\lambda_\tau [-(2\pi\alpha') \Pi_\mu G^{\mu\nu} B_{\nu\theta} + (2\pi\alpha') B_{\mu\theta} G^{\mu\nu} B_{\nu\theta} p_{\tilde{\theta}} + B_{\mu\theta} G^{\mu\theta} \partial_\sigma \tilde{\theta} - \\ &\quad - B_{\mu\theta} G^{\mu\theta} B_{\theta\nu} \partial_\sigma x^\nu + (2\pi\alpha') G_{\theta\theta} p_{\tilde{\theta}} - G_{\theta\nu} \partial_\sigma x^\nu] + \lambda_\sigma \partial_\sigma \tilde{\theta}. \end{aligned} \quad (11)$$

Then after some algebra we find

$$\begin{aligned} g^{\mu\nu} \hat{\Pi}_\nu &= \frac{1}{2(2\pi\alpha') \lambda_\tau} (\mathbf{X}^\mu + 2\lambda_\tau \mathbf{V}^\mu + 2(2\pi\alpha') \lambda_\tau G^{\mu\nu} B_{\nu\theta} p_{\tilde{\theta}}), \\ p_{\tilde{\theta}} &= \frac{1}{2(2\pi\alpha') \lambda_\tau G_{\theta\theta}} (\Theta + B_{\mu\theta} \mathbf{X}^\mu + 2\lambda_\tau G_{\mu\theta} \partial_\sigma x^\mu), \end{aligned} \quad (12)$$

where

$$\mathbf{V}^\mu = G^{\mu\theta} \partial_\sigma \tilde{\theta} - G^{\mu\theta} B_{\theta\nu} \partial_\sigma x^\nu, \quad \mathbf{X}^\mu = \partial_\tau x^\mu - \lambda_\sigma \partial_\sigma x^\mu, \quad \Theta = \partial_\tau \tilde{\theta} - \lambda_\sigma \partial_\sigma \tilde{\theta}. \quad (13)$$

In order to express $\hat{\Pi}_\mu$ from (12) we have to find the inverse matrix to $G^{\mu\nu}$. Recall that by definition

$$G_{MN} G^{NK} = \delta_M^K, \quad G_{\mu M} G^{N\mu} = G_{\mu\rho} G^{\rho N} + G_{\mu\theta} G^{\theta N} = \delta_\mu^N \quad (14)$$

and hence we see that $G_{\mu\nu}$ is not inverse to $G^{\mu\nu}$. It turns out that given matrix has the form

$$h_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu\theta} G_{\nu\theta}}{G_{\theta\theta}} \quad (15)$$

as can be easily seen from (14)

$$h_{\mu\nu} G^{\nu\rho} = \delta_\mu^\rho, \quad G^{\theta\mu} h_{\mu\nu} = -\frac{G_{\theta\nu}}{G_{\theta\theta}}. \quad (16)$$

Then we obtain

$$\hat{\Pi}_\mu = \frac{1}{2(2\pi\alpha') \lambda_\tau} h_{\mu\nu} (\mathbf{X}^\nu + 2\lambda_\tau \mathbf{V}^\nu + 2(2\pi\alpha') \lambda_\tau G^{\nu\rho} B_{\rho\theta} p_{\tilde{\theta}}) \quad (17)$$

and after some algebra we find the Lagrangian for dual theory in the form

$$\begin{aligned} \tilde{\mathcal{L}} &= p_{\tilde{\theta}} \partial_\tau \tilde{\theta} + p_\mu \partial_\sigma x^\mu - \lambda_\tau \tilde{\mathcal{H}}_\tau - \lambda_\sigma \tilde{\mathcal{H}}_\sigma = \\ &= \frac{1}{4(2\pi\alpha') \lambda_\tau} (\tilde{g}_{\tau\tau} + 2\lambda_\sigma \tilde{g}_{\tau\sigma} + \lambda_\sigma^2 \tilde{g}_{\sigma\sigma}) - \frac{1}{2\pi\alpha'} \lambda_\tau \tilde{g}_{\sigma\sigma} - \\ &\quad - \frac{1}{2\pi\alpha'} \partial_\tau x^\mu \tilde{B}_{\mu\nu} \partial_\sigma x^\nu - \frac{1}{2\pi\alpha'} \partial_\tau \tilde{\theta} \tilde{B}_{\tilde{\theta}\mu} \partial_\sigma x^\mu - \frac{1}{2\pi\alpha'} \partial_\tau x^\mu \tilde{B}_{\mu\tilde{\theta}} \partial_\sigma \tilde{\theta}, \end{aligned} \quad (18)$$

where

$$\tilde{g}_{\alpha\beta} = \tilde{G}_{\tilde{\theta}\tilde{\theta}}\partial_\alpha\tilde{\theta}\partial_\beta\tilde{\theta} + \tilde{G}_{\tilde{\theta}\mu}\partial_\alpha\tilde{\theta}\partial_\beta x^\mu + \tilde{G}_{\mu\tilde{\theta}}\partial_\alpha x^\mu\partial_\beta\tilde{\theta} + \tilde{G}_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu, \quad (19)$$

and where we defined T-dual components of the metric and NS-NS two form

$$\begin{aligned} \tilde{G}_{\mu\nu} &= h_{\mu\nu} + \frac{B_{\mu\theta}B_{\nu\theta}}{G_{\theta\theta}}, \quad \tilde{G}_{\mu\tilde{\theta}} = \tilde{G}_{\tilde{\theta}\mu} = \frac{B_{\mu\theta}}{G_{\theta\theta}}, \quad \tilde{G}_{\tilde{\theta}\tilde{\theta}} = \frac{1}{G_{\theta\theta}} \\ \tilde{B}_{\mu\nu} &= B_{\mu\nu} + \frac{B_{\mu\theta}G_{\theta\nu}}{G_{\theta\theta}} - \frac{G_{\mu\theta}B_{\theta\nu}}{G_{\theta\theta}}, \quad \tilde{B}_{\tilde{\theta}\mu} = -\tilde{B}_{\mu\tilde{\theta}} = -\frac{G_{\theta\mu}}{G_{\theta\theta}} \end{aligned} \quad (20)$$

that coincide with Buscher's transformations [14, 15]. In order to have more familiar form of the Lagrangian density we solve the equations of motion for λ_τ and λ_σ that follow from (18). Explicitly, the equation of motion for λ_σ has the form

$$\tilde{g}_{\tau\sigma} + \lambda_\sigma \tilde{g}_{\sigma\sigma} = 0 \quad (21)$$

while the equation of motion for λ_τ takes the form

$$-\frac{1}{4\lambda_\tau^2}[\tilde{g}_{\tau\tau} + 2\lambda_\sigma \tilde{g}_{\tau\sigma} + \lambda_\sigma^2 \tilde{g}_{\sigma\sigma}] + \tilde{g}_{\sigma\sigma} = 0 \quad (22)$$

that together with (21) implies

$$\lambda_\tau = \frac{\sqrt{\tilde{g}_{\tau\sigma}^2 - \tilde{g}_{\tau\tau}\tilde{g}_{\sigma\sigma}}}{2\tilde{g}_{\sigma\sigma}}. \quad (23)$$

Inserting these expressions into the Lagrangian density (18) we obtain the final result

$$\tilde{\mathcal{L}} = -\frac{1}{2\pi\alpha'} \left(\sqrt{-\det \tilde{g}} + \frac{1}{2}\epsilon^{\alpha\beta}\tilde{b}_{\alpha\beta} \right) \quad (24)$$

that is the Lagrangian density for Nambu-Gotto string in T-dual background.

3 T-duality and Double Wick rotation

We have shown in the previous section that canonical transformation in the Hamiltonian formulation of NG string gives the action for the string in T-dual background. Using this result we focus in this section on the relation between T-duality and double Wick rotation. We presume that the background has the form [7, 8, 9]

$$\begin{aligned} ds^2 &= G_{MN}dx^M dx^N = G_{tt}dt^2 + G_{\varphi\varphi}d\varphi^2 + G_{\mu\nu}dx^\mu dx^\nu, \\ B &= B_{MN}dX^M dX^N = B_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \quad (25)$$

where μ, ν denote the transverse directions. Following [7, 8] we introduce light cone coordinates

$$x^- = \varphi - t, \quad x^+ = (1-a)t + a\varphi \quad (26)$$

with inverse relations

$$t = x^+ - ax^-, \quad \varphi = x^+ + (1-a)x^-. \quad (27)$$

Then corresponding metric components have the form

$$G_{++} = G_{tt} + G_{\varphi\varphi}, \quad G_{--} = G_{tt}a^2 + (1-a)^2G_{\varphi\varphi}, \quad G_{+-} = -aG_{tt} + (1-a)G_{\varphi\varphi} \quad (28)$$

with inverse

$$\begin{aligned} G^{++} &= \frac{G_{tt}a^2 + (1-a)^2G_{\varphi\varphi}}{G_{tt}G_{\varphi\varphi}}, \quad G^{--} = \frac{G_{tt} + G_{\varphi\varphi}}{G_{tt}G_{\varphi\varphi}}, \\ G^{+-} &= \frac{aG_{tt} - (1-a)G_{\varphi\varphi}}{G_{tt}G_{\varphi\varphi}}. \end{aligned} \quad (29)$$

In the light cone coordinates the Hamiltonian and diffeomorphism constraints have the form

$$\begin{aligned} \mathcal{H}_\tau &= (2\pi\alpha')p_+G^{++}p_+ + 2(2\pi\alpha')p_+G^{+-}p_- + (2\pi\alpha')p_-G^{--}p_- + \\ &+ \frac{1}{2\pi\alpha'}[G_{++}(\partial_\sigma x^+)^2 + 2G_{+-}\partial_\sigma x^+\partial_\sigma x^- + G_{--}(\partial_\sigma x^-)^2] + \mathcal{H}_x, \\ \mathcal{H}_\sigma &= p_+\partial_\sigma x^+ + p_-\partial_\sigma x^- + p_\mu\partial_\sigma x^\mu, \end{aligned} \quad (30)$$

where

$$\mathcal{H}_x = (2\pi\alpha')\Pi_\mu G^{\mu\nu}\Pi_\nu + \frac{1}{2\pi\alpha'}G_{\mu\nu}\partial_\sigma x^\mu\partial_\sigma x^\nu, \quad \Pi_\mu = p_\mu + \frac{1}{2\pi\alpha'}B_{\mu\nu}\partial_\sigma x^\nu. \quad (31)$$

Now we are ready to study the relation between T-duality and double Wick rotation. Let us presume that the background does not depend on x^- . Our goal is to perform the canonical transformation from x^- to ψ , where, following discussion presented in previous section, the generating function has the form

$$G(\theta, \psi) = \frac{1}{4\pi\alpha'} \int d\sigma (\partial_\sigma x^- \psi - x^- \partial_\sigma \psi). \quad (32)$$

Let us denote the momentum conjugate to ψ as p_ψ . Then we obtain

$$p_\psi = -\frac{1}{2\pi\alpha'}\partial_\sigma x^-, \quad p_- = -\frac{1}{2\pi\alpha'}\partial_\sigma \psi \quad (33)$$

so that we obtain T-dual Hamiltonian when we replace $\partial_\sigma x^-$ with $-(2\pi\alpha')p_\psi$ and p_- with $-\frac{1}{2\pi\alpha'}\partial_\sigma\psi$ and hence

$$\begin{aligned}\tilde{\mathcal{H}}_\tau &= (2\pi\alpha')p_+G^{++}p_+ - 2p_+G^{+-}\partial_\sigma\psi + \frac{1}{2\pi\alpha'}G^{--}(\partial_\sigma\psi)^2 + \frac{1}{2\pi\alpha'}(G_{++}(\partial_\sigma x^+)^2 + \\ &\quad - 4\pi\alpha'G_{+-}\partial_\sigma x^+p_\psi + (2\pi\alpha')^2G_{--}(p_\psi)^2) + \mathcal{H}_x, \\ \tilde{\mathcal{H}}_\sigma &= p_+\partial_\sigma x^+ + p_\psi\partial_\sigma\psi + p_\mu\partial_\sigma x^\mu.\end{aligned}\tag{34}$$

Then following [9] we perform analytic continuation in the target space-time

$$(x^+, \psi) \rightarrow (i\tilde{\psi}, -i\tilde{x}^+), \quad (p_+, p_\psi) \rightarrow (-i\tilde{p}_\psi, i\tilde{p}_+).\tag{35}$$

so that we obtain

$$\begin{aligned}\tilde{\mathcal{H}}_\tau &= -(2\pi\alpha')\tilde{p}_\psi G^{++}\tilde{p}_\psi + 2\tilde{p}_\psi G^{+-}\partial_\sigma\tilde{x}^+ - \frac{1}{2\pi\alpha'}G^{--}(\partial_\sigma\tilde{x}^+)^2 + \\ &\quad + \frac{1}{2\pi\alpha'}\left(-G_{++}(\partial_\sigma\tilde{\psi})^2 + 4\pi\alpha'G_{+-}\partial_\sigma\tilde{\psi}\tilde{p}_+ - (2\pi\alpha')^2G_{--}(\tilde{p}_+)^2\right) + \mathcal{H}_x, \\ \tilde{\mathcal{H}}_\sigma &= p_+\partial_\sigma x^+ + \tilde{p}_\psi\partial_\sigma\psi + p_\mu\partial_\sigma x^\mu.\end{aligned}\tag{36}$$

Note that the way how the conjugate momenta p_+, p_ψ transform under the analytic continuation is given by the requirement that all terms in $\tilde{\mathcal{H}}_\sigma$ come with + sign since we demand that the string theory is invariant under world-sheet diffeomorphism and hence Wick rotated $\tilde{\mathcal{H}}_\sigma$ should have the same form as the original one.

Finally we perform T-duality transformation along $\tilde{\psi}$ direction that gives

$$\tilde{p}_\psi = -\frac{1}{2\pi\alpha'}\partial_\sigma\tilde{\phi}, \quad p_{\tilde{\phi}} = -\frac{1}{2\pi\alpha'}\partial_\sigma\tilde{\psi}\tag{37}$$

so that we obtain the final form of the Hamiltonian and diffeomorphism constraints

$$\begin{aligned}\tilde{\mathcal{H}}_\tau &= -\frac{1}{2\pi\alpha'}(\partial_\sigma\tilde{\phi})^2G^{++} - \frac{2}{2\pi\alpha'}\partial_\sigma\tilde{\phi}G^{+-}\partial_\sigma\tilde{x}^+ - \frac{1}{2\pi\alpha'}G^{--}(\partial_\sigma\tilde{x}^+)^2 + \\ &\quad + (2\pi\alpha')\left(-G_{++}p_{\tilde{\phi}}^2 - 2G_{+-}p_{\tilde{\phi}}\tilde{p}_+ - G_{--}(\tilde{p}_+)^2\right) + \mathcal{H}_x, \\ \tilde{\mathcal{H}}_\sigma &= p_+\partial_\sigma x^+ + \tilde{p}_\psi\partial_\sigma\psi + p_\mu\partial_\sigma x^\mu.\end{aligned}\tag{38}$$

In order to find the Hamiltonian for the physical degrees of freedom we have to fix the gauge. It turns out that it is natural to use uniform gauge fixing

$$p_{\tilde{\phi}} = \frac{1}{2\pi\alpha'}, \quad x^+ = \tau.\tag{39}$$

Then from $\tilde{\mathcal{H}}_\sigma = 0$ we find

$$\partial_\sigma\tilde{\phi} = -(2\pi\alpha')(p_\mu\partial_\sigma x^\mu)\tag{40}$$

and hence the Hamiltonian constraint is equal to

$$\tilde{\mathcal{H}}_\tau = -(2\pi\alpha')(\partial_\sigma x^\mu p_\mu)^2 G^{++} - \frac{1}{2\pi\alpha'} G_{++} - 2G_{+-}\tilde{p}_+ - (2\pi\alpha')\tilde{p}_+ G_{--}\tilde{p}_+ + \mathcal{H}_x = 0 . \quad (41)$$

Note that due to the gauge fixing the constraints $\tilde{\mathcal{H}}_\tau, \tilde{\mathcal{H}}_\sigma$ vanish strongly and hence (41) serves as the quadratic equation for \tilde{p}_+ . In fact, $-\tilde{p}_+$ should be identified as the Hamiltonian density for the physical degrees of freedom after gauge fixing.

Now we would like to compare the equation (41) with the equation that defines the Hamiltonian density for the physical degrees of freedom for of the uniform gauge fixed string. Note that this gauge is imposed in the Hamiltonian formulation of the string we identify $p_- = \frac{1}{2\pi\alpha'}$. Equivalently we can impose given gauge in T-dual theory when we identify ψ with σ [16]. This construction has an advantage since it does not require to go to the Hamiltonian formulation of given theory which could be extremely difficult in case of Green-Schwarz action. However in our case we can either choose the Hamiltonian constraint (30) and impose the gauge $p_- = \frac{1}{2\pi\alpha'}, x^+ = \tau$ or use T-dual Hamiltonian constraint (34) with the following gauge fixing functions

$$\psi = \sigma, x^+ = \tau . \quad (42)$$

We choose the second possibility and using (42) in \mathcal{H}_σ we obtain $p_\psi = -(\partial_\sigma x^\mu p_\mu)$ that together with (42) implies that (34) has the form

$$\mathcal{H}_\tau = (2\pi\alpha')p_+ G^{++} p_+ - 2p_+ G^{+-} + \frac{1}{2\pi\alpha'} G^{--} + (2\pi\alpha') G_{--} (p_\mu \partial_\sigma x^\mu)^2 + \mathcal{H}_x = 0 . \quad (43)$$

This is quadratic equation for p_+ . Comparing (43) with (41) we see that they have the same form when we define metric components

$$\tilde{G}^{++} = -G_{--}, \quad \tilde{G}^{--} = -G_{++}, \quad \tilde{G}^{+-} = G_{+-} . \quad (44)$$

In other words, the sequence of T-dualities and analytic continuation in target space-time implies the transformation of the components of the target metric that has the same form as the double Wick rotation in the uniform gauge fixed bosonic string [9]. On the other hand we also see that components of NS-NS two form that are transverse to the directions where these dualities were performed do not transform. At this place we see the difference with the double Wick rotation of the gauge fixed action [9] since in this case these components change the sign. In order to resolve this issue we can extend the sequence of T-duality transformation and Wick rotation with an additional transformation $B \rightarrow -B$ since as we argued in the introduction whenever the background field B is solution of the supergravity equations of motion so $-B$ is too. In other words we have the equivalence between world-sheet double Wick rotation and the sequence: "T-duality-target space Wick rotation-T-duality- $B \rightarrow -B$."

4 D1-brane and Duality Transformation

In this section we apply the same ideas to the case of DBI and WZ action for D1-brane. Let us start with D1-brane action

$$\begin{aligned} S = & -T_{D1} \int d\tau d\sigma e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + b_{\alpha\beta} + (2\pi\alpha')F_{\alpha\beta})} + \\ & + T_{D1} \int d\tau d\sigma [C^{(0)}(b_{\tau\sigma} + (2\pi\alpha')F_{\tau\sigma}) + C_{\tau\sigma}^{(2)}] , \end{aligned} \quad (45)$$

where

$$g_{\alpha\beta} = G_{MN} \partial_\alpha x^M \partial_\beta x^N , \quad b_{\alpha\beta} = B_{MN} \partial_\alpha x^M \partial_\beta x^N , \quad C_{\tau\sigma}^{(2)} = C_{MN}^{(2)} \partial_\tau x^M \partial_\sigma x^N , \quad (46)$$

and where $x^M(\tau, \sigma)$ are embedding coordinates for D1-brane in given background. Further, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength of the world-volume gauge field A_α , $\alpha = \tau, \sigma$. Finally T_{D1} is D1-brane tension $T_{D1} = \frac{1}{2\pi\alpha'}$.

Before we proceed to the Hamiltonian formulation of the action (45) it is useful to use following formula

$$\det(g_{\alpha\beta} + b_{\alpha\beta} + (2\pi\alpha')F_{\alpha\beta}) = \det g + (b_{\tau\sigma} + (2\pi\alpha')F_{\tau\sigma})^2 \quad (47)$$

that holds in two dimensions only. Then from the action (45) we find momenta conjugate to x^M, A_σ and A_τ respectively

$$\begin{aligned} p_M &= T_{D1} \frac{e^{-\Phi}}{\sqrt{-\det g - ((2\pi\alpha')F_{\tau\sigma} + b_{\tau\sigma})^2}} (G_{MN} \partial_\alpha x^N g^{\alpha\tau} \det g + \\ &+ ((2\pi\alpha')F_{\tau\sigma} + b_{\tau\sigma}) B_{MN} \partial_\sigma x^N) + T_{D1} (C^{(0)} B_{MN} \partial_\sigma x^N + C_{MN}^{(2)} \partial_\sigma x^N) , \\ \pi^\sigma &= \frac{e^{-\Phi} T_{D1} (2\pi\alpha') ((2\pi\alpha')F_{\tau\sigma} + b_{\tau\sigma})}{\sqrt{-\det g - ((2\pi\alpha')F_{\tau\sigma} + b_{\tau\sigma})^2}} + T_{D1} (2\pi\alpha') C^{(0)} , \quad \pi^\tau \approx 0 . \end{aligned} \quad (48)$$

Using these relations we find that the bare Hamiltonian is equal to

$$H_B = \int d\sigma (p_M \partial_\tau x^M + \pi^\sigma \partial_\tau A_\sigma - \mathcal{L}) = \int d\sigma \pi^\sigma \partial_\sigma A_\tau \quad (49)$$

while we have three primary constraints

$$\begin{aligned} \mathcal{H}_\sigma &\equiv p_M \partial_\sigma x^M \approx 0 , \quad \pi^\tau \approx 0 , \\ \mathcal{H}_\tau &\equiv (2\pi\alpha') \Pi_M G^{MN} \Pi_N + \frac{1}{2\pi\alpha'} \left(e^{-2\Phi} + (\pi^\sigma - C^{(0)})^2 \right) G_{MN} \partial_\sigma x^M \partial_\sigma x^N , \end{aligned} \quad (50)$$

where

$$\Pi_M \equiv p_M - \frac{\pi^\sigma}{2\pi\alpha'} B_{MN} \partial_\sigma x^N - \frac{1}{2\pi\alpha'} C_{MN}^{(2)} \partial_\sigma x^N , \quad (51)$$

and where we used the fact that $T_{D1} = \frac{1}{2\pi\alpha'}$. According to the standard treatment of the constraint systems we introduce the extended Hamiltonian with all primary constraints included

$$H = \int d\sigma (\lambda_\tau \mathcal{H}_\tau + \lambda_\sigma \mathcal{H}_\sigma - A_\tau \partial_\sigma \pi^\sigma + v_\tau \pi^\tau) , \quad (52)$$

where $\lambda_{\tau,\sigma}$ and v_τ are Lagrange multipliers corresponding to the constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$ and π^τ .

Now the requirement of the preservation of the primary constraint $\pi^\tau \approx 0$ implies the secondary constraint

$$\mathcal{G} = \partial_\sigma \pi^\sigma \approx 0 . \quad (53)$$

Then it can be shown that $\mathcal{H}_\tau, \mathcal{H}_\sigma$ are the first class constraints, for details, see [17].

Now we can formally proceed to the discussion of the canonical transformation as in the case of fundamental string. Let us presume that there is a direction that is invariant under constant shift

$$\theta \rightarrow \theta + \epsilon , \quad \epsilon = \text{const} . \quad (54)$$

Then we again consider the generating function into the form

$$G(\theta, \tilde{\theta}) = \frac{1}{4\pi\alpha'} \int d\sigma (\partial_\sigma \theta \tilde{\theta} - \theta \partial_\sigma \tilde{\theta}) \quad (55)$$

that implies the relation between momenta p_θ and $p_{\tilde{\theta}}$ respectively

$$p_{\tilde{\theta}} = -\frac{1}{2\pi\alpha'} \partial_\sigma \theta , \quad p_\theta = -\frac{1}{2\pi\alpha'} \partial_\sigma \tilde{\theta} . \quad (56)$$

Then we obtain dual Hamiltonian when we replace $\partial_\sigma \theta$ with $-(2\pi\alpha')p_{\tilde{\theta}}$ and p_θ with $-\frac{1}{2\pi\alpha'}\partial_\sigma \tilde{\theta}$ so that

$$\begin{aligned} \tilde{\mathcal{H}}_\tau &= (2\pi\alpha')\Pi_\theta G^{\theta\theta}\Pi_\theta + 2(2\pi\alpha')\Pi_\theta G^{\theta\mu}\Pi_\mu + (2\pi\alpha')\Pi_\nu G^{\mu\nu}\Pi_\nu + \\ &+ \frac{1}{2\pi\alpha'} \left(e^{-2\Phi} + (\pi^\sigma - C^{(0)})^2 \right) G_{\mu\nu} \partial_\sigma x^\mu \partial_\sigma x^\nu - \\ &- 2 \left(e^{-2\Phi} + (\pi^\sigma - C^{(0)})^2 \right) G_{\mu\theta} \partial_\sigma x^\mu p_{\tilde{\theta}} + \\ &+ (2\pi\alpha') \left(e^{-2\Phi} + (\pi^\sigma - C^{(0)})^2 \right) G_{\theta\theta} p_{\tilde{\theta}} p_{\tilde{\theta}} , \\ \tilde{\mathcal{H}}_\sigma &= p_\mu \partial_\sigma x^\mu + p_{\tilde{\theta}} \partial_\sigma \tilde{\theta} . \end{aligned} \quad (57)$$

In order to see how the background fields transform under this duality transformation we should find corresponding Lagrangian. Before we proceed to this question we should stress one important point. In principle the electric flux that is given in (57) is off-shell. However we know that this electric flux is proportional to the number of the fundamental strings. Our goal is to compare actions where this number is

the same so that we consider canonical transformations for D1-brane theory where we fix the gauge symmetry so that $\pi^\sigma = \text{const.}$ In fact, if π^σ were the dynamical variable we would get very complicated form of the Lagrangian density due to the fact that now the Hamiltonian contains term like $(\pi^\sigma)^2 p_{\tilde{\theta}}^2$.

In order to proceed to the Lagrangian formulation we introduce following notations

$$\begin{aligned}\Pi_\mu &= \hat{\Pi}_\mu + \mathbf{V}_\mu p_{\tilde{\theta}} , \quad \mathbf{V}_\mu = \pi^\sigma B_{\mu\theta} + C_{\mu\theta}^{(2)} , \\ \hat{\Pi}_\mu &= p_\mu - \frac{\pi^\sigma}{2\pi\alpha'} B_{\mu\nu} \partial_\sigma x^\nu - \frac{1}{2\pi\alpha'} C_{\mu\nu}^{(2)} \partial_\sigma x^\nu , \\ \Pi_\theta &= -\frac{1}{2\pi\alpha'} (\partial_\sigma \tilde{\theta} + \mathbf{V}_\mu \partial_\sigma x^\mu) , \quad \mathbf{X} = e^{-2\Phi} + (\pi^\sigma - C^{(0)})^2 .\end{aligned}\tag{58}$$

so that we can write $\tilde{\mathcal{H}}_\tau$ in the form

$$\begin{aligned}\tilde{\mathcal{H}}_\tau &= (2\pi\alpha') p_{\tilde{\theta}} (\mathbf{V}_\mu G^{\mu\nu} \mathbf{V}_\nu + \mathbf{X} G_{\theta\theta}) p_{\tilde{\theta}} + \\ &+ (2\pi\alpha') \hat{\Pi}_\mu G^{\mu\nu} \hat{\Pi}_\nu + \hat{\Pi}_\mu (-2\partial_\sigma \tilde{\theta} G^{\theta\mu} - 2\mathbf{V}_\nu \partial_\sigma x^\nu G^{\theta\mu}) + 4\pi\alpha' \hat{\Pi}_\mu G^{\mu\nu} \mathbf{V}_\nu p_{\tilde{\theta}} + \\ &+ p_{\tilde{\theta}} (-2\partial_\sigma \tilde{\theta} G^{\theta\mu} \mathbf{V}_\mu - 2\mathbf{V}_\nu \partial_\sigma x^\nu G^{\theta\mu} \mathbf{V}_\mu + 2\mathbf{X} G_{\mu\theta} \partial_\sigma x^\mu) + \\ &+ \frac{1}{2\pi\alpha'} \left(G^{\theta\theta} (\partial_\sigma \tilde{\theta})^2 - 2\mathbf{V}_\mu \partial_\sigma x^\mu G^{\theta\theta} \partial_\sigma \tilde{\theta} + \partial_\sigma x^\mu (\mathbf{V}_\mu G^{\theta\theta} \mathbf{V}_\nu) \partial_\sigma x^\nu + \mathbf{X} G_{\mu\nu} \partial_\sigma x^\mu \partial_\sigma x^\nu \right)\end{aligned}\tag{59}$$

and hence we obtain

$$\begin{aligned}\partial_\tau \tilde{\theta} &= \left\{ \tilde{\theta}, H \right\} = 4\pi\alpha' \lambda_\tau (\mathbf{V}_\mu G^{\mu\nu} \mathbf{V}_\nu + \mathbf{X}) G_{\theta\theta} p_{\tilde{\theta}} + \\ &+ 4\pi\alpha' \hat{\Pi}_\mu G^{\mu\nu} \mathbf{V}_\nu - 2(\partial_\sigma \tilde{\theta} G^{\theta\mu} \mathbf{V}_\nu + \mathbf{V}_\sigma \partial_\sigma x^\sigma G^{\theta\mu} \mathbf{V}_\mu - \mathbf{X} G_{\mu\theta} \partial_\sigma x^\mu) + \lambda_\sigma \partial_\sigma \theta , \\ \partial_\tau x^\mu &= \{x^\mu, H\} = 2\lambda_\tau ((2\pi\alpha') G^{\mu\nu} \hat{\Pi}_\mu - (\partial_\sigma \tilde{\theta} + \mathbf{V}_\nu \partial_\sigma x^\nu) G^{\theta\mu} + (2\pi\alpha') G^{\mu\nu} \mathbf{V}_\nu p_{\tilde{\theta}}) + \lambda_\sigma \partial_\sigma x^\mu .\end{aligned}\tag{60}$$

Now from the last equation we get

$$\hat{\Pi}_\mu = \frac{1}{2\lambda_\tau} h_{\mu\nu} \mathbf{X}^\nu - \frac{G_{\mu\theta}}{G_{\theta\theta}} (\partial_\sigma \tilde{\theta} + \mathbf{V}_\nu \partial_\sigma x^\nu) - \mathbf{V}_\mu p_{\tilde{\theta}} ,\tag{61}$$

where

$$p_{\tilde{\theta}} = \frac{1}{4\pi\alpha' \lambda_\tau G_{\theta\theta} \mathbf{X}} (\Theta + \mathbf{X}^\mu \mathbf{V}_\mu + 2\lambda_\tau \mathbf{X} G_{\mu\theta} \partial_\sigma x^\mu) ,\tag{62}$$

and where Θ, \mathbf{X}^μ are defined in (13). If we then proceed in the same way as in case

of the fundamental string we derive final form of the dual Lagrangian density

$$\begin{aligned}
\tilde{\mathcal{L}} &= \partial_\tau x^\mu p_\mu + \partial_\tau \tilde{\theta} p_{\tilde{\theta}} - H = \\
&= T_{D1} \frac{1}{4\lambda_\tau} (\tilde{g}_{\tau\tau} - 2\lambda_\sigma \tilde{g}_{\tau\sigma} + \lambda_\sigma^2 \tilde{g}_{\sigma\sigma}) - T_{D1} \lambda_\tau \mathbf{X} \tilde{g}_{\sigma\sigma} + \\
&+ T_{D1} \left(\pi^\sigma B_{\mu\nu} + C_{\mu\nu}^{(2)} - \frac{G_{\mu\theta}}{G_{\theta\theta}} (\pi^\sigma B_{\mu\theta} + C_{\nu\theta}^{(2)}) + \frac{G_{\nu\theta}}{G_{\theta\theta}} (\pi^\sigma B_{\mu\theta} + C_{\mu\theta}^{(2)}) \partial_\tau x^\mu \partial_\sigma x^\nu + \right. \\
&+ \left. \frac{G_{\mu\theta}}{G_{\theta\theta}} \partial_\tau \tilde{\theta} \partial_\sigma x^\mu - \frac{G_{\mu\theta}}{G_{\theta\theta}} \partial_\tau x^\mu \partial_\sigma \tilde{\theta} \right) ,
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\tilde{g}_{\tau\tau} &= \partial_\tau x^\mu \left(h_{\mu\nu} + \frac{\mathbf{V}_\mu \mathbf{V}_\nu}{G_{\theta\theta} \mathbf{X}} \right) \partial_\tau x^\nu + \frac{1}{G_{\theta\theta} \mathbf{X}} (\partial_\tau \tilde{\theta})^2 + \frac{2}{G_{\theta\theta} \mathbf{X}} \partial_\tau \tilde{\theta} \partial_\tau x^\mu \mathbf{V}_\mu , \\
\tilde{g}_{\tau\sigma} &= \partial_\tau x^\mu \left(h_{\mu\nu} + \frac{\mathbf{V}_\mu \mathbf{V}_\nu}{G_{\theta\theta} \mathbf{X}} \right) \partial_\sigma x^\nu + \frac{1}{G_{\theta\theta} \mathbf{X}} \partial_\tau \tilde{\theta} \partial_\sigma \tilde{\theta} + \frac{2}{G_{\theta\theta} \mathbf{X}} \partial_\tau \tilde{\theta} \partial_\sigma x^\mu \mathbf{V}_\mu + \frac{2}{G_{\theta\theta} \mathbf{X}} \partial_\sigma \tilde{\theta} \partial_\tau x^\mu \mathbf{V}_\mu , \\
\tilde{g}_{\sigma\sigma} &= \partial_\sigma x^\mu \left(h_{\mu\nu} + \frac{\mathbf{V}_\mu \mathbf{V}_\nu}{G_{\theta\theta} \mathbf{X}} \right) \partial_\sigma x^\nu + \frac{2}{G_{\theta\theta} \mathbf{X}} \partial_\tau \tilde{\theta} \partial_\sigma x^\mu \mathbf{V}_\mu + \frac{1}{G_{\theta\theta} \mathbf{X}} (\partial_\sigma \tilde{\theta})^2 .
\end{aligned} \tag{64}$$

Finally we eliminate $\lambda_\tau, \lambda_\sigma$ using corresponding equations of motion

$$\lambda_\sigma = -\frac{\tilde{g}_{\tau\sigma}}{\tilde{g}_{\sigma\sigma}} , \quad \lambda_\tau = \frac{1}{2\sqrt{\mathbf{X}} \tilde{g}_{\sigma\sigma}} \sqrt{\tilde{g}_{\tau\sigma}^2 - \tilde{g}_{\tau\tau} \tilde{g}_{\sigma\sigma}} . \tag{65}$$

Inserting back to the Lagrangian (63) we obtain final form of the dual Lagrangian density

$$\begin{aligned}
\mathcal{L} &= -T_{D1} \sqrt{e^{-2\Phi} + (\pi - C^{(0)})^2} \sqrt{-\det \tilde{g}_{\alpha\beta}} + \\
&+ T_{D1} \left((\pi \tilde{B}_{\mu\nu} + \tilde{C}_{\mu\nu}^{(2)}) \partial_\tau x^\mu \partial_\sigma x^\nu + \frac{G_{\mu\theta}}{G_{\theta\theta}} \partial_\tau \tilde{\theta} \partial_\sigma x^\mu - \frac{G_{\mu\theta}}{G_{\theta\theta}} \partial_\tau x^\mu \partial_\sigma \tilde{\theta} \right) ,
\end{aligned} \tag{66}$$

where

$$\begin{aligned}
\tilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu\theta} G_{\nu\theta}}{G_{\theta\theta}} + \frac{(\pi B_{\mu\theta} + C_{\mu\theta}^{(2)})(\pi B_{\nu\theta} + C_{\nu\theta}^{(2)})}{G_{\theta\theta}(e^{-2\Phi} + (\pi - C^{(0)})^2)} , \\
\tilde{G}_{\tilde{\theta}\tilde{\theta}} &= \frac{1}{G_{\theta\theta}(e^{-2\Phi} + (\pi - C^{(0)})^2)} , \\
\tilde{G}_{\tilde{\theta}\mu} &= \tilde{G}_{\mu\tilde{\theta}} = \frac{1}{G_{\theta\theta}(e^{-2\Phi} + (\pi - C^{(0)})^2)} (\pi B_{\mu\theta} + C_{\mu\theta}^{(2)}) , \\
\tilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{G_{\mu\theta} B_{\nu\theta}}{G_{\theta\theta}} + \frac{G_{\nu\theta} B_{\mu\theta}}{G_{\theta\theta}} , \\
\tilde{C}_{\mu\nu}^{(2)} &= C_{\mu\nu}^{(2)} - \frac{G_{\mu\theta} C_{\nu\theta}^{(2)}}{G_{\theta\theta}} + \frac{G_{\nu\theta} C_{\mu\theta}^{(2)}}{G_{\theta\theta}} ,
\end{aligned} \tag{67}$$

We would like to give physical interpretation of given transformation rules. To begin with note that Type IIB theory is invariant under $SL(2, Z)$ symmetry

$$\begin{aligned}\hat{G}_{MN} &= e^{\frac{1}{2}(\hat{\Phi}-\Phi)} G_{MN} , \quad \hat{\tau} = \frac{p\tau + q}{r\tau + s} , \\ \hat{B}_{MN} &= sB_{MN} - rC_{MN}^{(2)} , \quad \hat{C}_{MN}^{(2)} = pC_{MN} - qB_{MN} ,\end{aligned}\tag{68}$$

where $\tau = C^{(0)} + ie^{-\Phi}$ and where $ps - qr = 1$. Let us presume that $\pi = -|\pi|$ and then choose following values of the parameters p, q, r, s [18] :

$$p = 0 , q = -1 , r = 1 , s = |\pi| \tag{69}$$

so that we explicitly obtain

$$\begin{aligned}\hat{C}^{(0)} &= -\frac{C^{(0)} + |\pi|}{(C^{(0)} + |\pi|)^2 + e^{-2\Phi}} , \quad e^{-\hat{\Phi}} = \frac{e^{-\Phi}}{(C^{(0)} + |\pi|)^2 + e^{-2\Phi}} , \\ \hat{G}_{MN} &= \sqrt{(C^{(0)} + |\pi|)^2 + e^{-2\Phi}} G_{MN} , \\ \hat{B}_{MN} &= |\pi| B_{MN} - C_{MN}^{(2)} , \quad \hat{C}_{MN}^{(2)} = B_{MN} .\end{aligned}\tag{70}$$

We see that when we combine the square root $\sqrt{(C^{(0)} + |\pi|)^2 + e^{-2\Phi}}$ with $\sqrt{-\det \tilde{g}}$ and use (70) we obtain that the Lagrangian density (66) has the form

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \sqrt{-\det \tilde{g}} + \frac{1}{2\pi\alpha'} \left(\bar{B}_{\mu\nu} \partial_\tau x^\mu \partial_\sigma x^\nu + \bar{B}_{\tilde{\theta}\mu} \partial_\tau \tilde{\theta} \partial_\sigma x^\mu + \bar{B}_{\mu\tilde{\theta}} \partial_\tau x^\mu \partial_\sigma \tilde{\theta} \right) , \tag{71}$$

where

$$\bar{g}_{\alpha\beta} = \bar{G}_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + \bar{G}_{\mu\tilde{\theta}} \partial_\alpha x^\mu \partial_\beta \tilde{\theta} + \bar{G}_{\tilde{\theta}\nu} \partial_\alpha \tilde{\theta} \partial_\beta x^\nu + \bar{G}_{\tilde{\theta}\tilde{\theta}} \partial_\alpha \tilde{\theta} \partial_\beta \tilde{\theta} , \tag{72}$$

and where the background fields $\bar{G}_{MN}, \bar{B}_{MN}$ have explicit form

$$\begin{aligned}\bar{G}_{\mu\nu} &= \hat{G}_{\mu\nu} - \frac{\hat{G}_{\mu\theta} \hat{G}_{\nu\theta}}{\hat{G}_{\theta\theta}} + \frac{\hat{B}_{\mu\theta} \hat{B}_{\nu\theta}}{\hat{G}_{\theta\theta}} , \\ \bar{G}_{\tilde{\theta}\tilde{\theta}} &= \frac{1}{\hat{G}_{\theta\theta}} , \quad \bar{G}_{\tilde{\theta}\mu} = \bar{G}_{\mu\tilde{\theta}} = \frac{1}{\hat{G}_{\theta\theta}} \hat{B}_{\mu\theta} , \\ \bar{B}_{\mu\nu} &= \hat{B}_{\mu\nu} - \frac{\hat{G}_{\mu\theta} \hat{B}_{\nu\theta}}{\hat{G}_{\theta\theta}} + \frac{\hat{G}_{\nu\theta} \hat{B}_{\mu\theta}}{\hat{G}_{\theta\theta}} , \\ \bar{B}_{\tilde{\theta}\mu} &= -\hat{B}_{\mu\tilde{\theta}} = \frac{\hat{G}_{\mu\theta}}{\hat{G}_{\theta\theta}} .\end{aligned}\tag{73}$$

Now the physical interpretation of the canonical transformation (55) on the world-volume of D1-brane is clear. It corresponds to the T-duality rules for the fundamental string moving in the S-dual background (70).

5 Canonical Transformations and Double Wick Rotation on D1-brane

In this section we perform the same analysis as in section (3) in order to find the relation between sequence of canonical transformations and Wick rotation with the double Wick rotation on the world-volume of gauge fixed D1-brane action. We again consider the metric and NS-NS two form field in the form (25) and we introduce the light cone coordinates as in (26) so that the metric components are given in (28) and (29). Further, using the relation between light-cone coordinates and the original ones we obtain following components of Ramond-Ramond two form

$$\begin{aligned} C_{+-}^{(2)} &= -C_{-+}^{(2)} = C_{t\varphi}^{(2)} , \\ C_{+\mu}^{(2)} &= -C_{\mu+}^{(2)} = C_{t\mu}^{(2)} + C_{\varphi\mu}^{(2)} , \\ C_{-\mu}^{(2)} &= -C_{\mu-}^{(2)} = -C_{t\mu}^{(2)} + (1-a)C_{\varphi\mu}^{(2)} . \end{aligned} \tag{74}$$

In the light cone coordinates the Hamiltonian and diffeomorphism constraints have the form

$$\begin{aligned} \mathcal{H}_\tau &= (2\pi\alpha')\Pi_+ G^{++}\Pi_+ + 2(2\pi\alpha')\Pi_+ G^{+-}\Pi_- + (2\pi\alpha')\Pi_- G^{--}\Pi_- + \\ &+ \frac{1}{2\pi\alpha'}\mathbf{X} \left(G_{++}(\partial_\sigma x^+)^2 + 2G_{+-}\partial_\sigma x^+ \partial_\sigma x^- + G_{--}(\partial_\sigma x^-)^2 \right) + \mathcal{H}_x , \\ \mathcal{H}_\sigma &= p_+ \partial_\sigma x^+ + p_- \partial_\sigma x^- + p_\mu \partial_\sigma x^\mu , \end{aligned} \tag{75}$$

where

$$\mathcal{H}_x \equiv (2\pi\alpha')\Pi_\mu G^{\mu\nu}\Pi_\nu + \frac{1}{2\pi\alpha'}\mathbf{X}\partial_\sigma x^\mu G_{\mu\nu}\partial_\sigma x^\nu , \tag{76}$$

and where $\mathbf{X} = \left(e^{-2\Phi} + (C^{(0)})^2 \right)$. Note that for simplicity we presume that there is no electric flux so that $\pi^\sigma = 0$. Now we are ready to study the relation between sequence of canonical transformation, target space Wick rotation and world-sheet double Wick rotation in the same way as in section (3). Let us presume that the background does not depend on x^- and perform the canonical transformation from x^- to ψ so we obtain the relation

$$p_\psi = -\frac{1}{2\pi\alpha'}\partial_\sigma x^- , \quad p_- = -\frac{1}{2\pi\alpha'}\partial_\sigma \psi \tag{77}$$

and hence we find

$$\begin{aligned}
\Pi_+ &= p_+ - \frac{1}{2\pi\alpha'} C_{+\mu}^{(2)} \partial_\sigma x^\mu + C_{+-}^{(2)} p_\psi , \\
\Pi_- &= -\frac{1}{2\pi\alpha'} (\partial_\sigma \psi + C_{-+}^{(2)} \partial_\sigma x^+) - \frac{1}{2\pi\alpha'} C_{-\mu}^{(2)} \partial_\sigma x^\mu , \\
\Pi_\mu &= p_\mu - \frac{1}{2\pi\alpha'} (C_{\mu\nu}^{(2)} \partial_\sigma x^\nu + C_{\mu+}^{(2)} \partial_\sigma x^+) + C_{\mu-}^{(2)} p_\psi .
\end{aligned} \tag{78}$$

As the next step we perform analytic continuation

$$(x^+, \psi) \rightarrow (i\tilde{\psi}, -i\tilde{x}^+) , \quad (p_+, p_\psi) \rightarrow (-i\tilde{p}_\psi, i\tilde{p}_+) \tag{79}$$

that implies

$$\begin{aligned}
\Pi_+ &\rightarrow -i(\tilde{p}_\psi - T_{D1} C_{+-}^{(2)} \tilde{p}_+) - \frac{1}{2\pi\alpha'} C_{+\mu}^{(2)} \partial_\sigma x^\mu , \\
\Pi_- &\rightarrow i(\partial_\sigma \tilde{x}^+ - T_{D1} C_{-+}^{(2)} \partial_\sigma \tilde{\psi}) - \frac{1}{2\pi\alpha'} C_{-\mu}^{(2)} \partial_\sigma x^\mu , \\
\Pi_\mu &\rightarrow p_\mu - T_{D1} C_{\mu\nu}^{(2)} \partial_\sigma x^\nu - iT_{D1} C_{\mu+}^{(2)} \partial_\sigma \tilde{\psi} + iT_{D1} C_{\mu-}^{(2)} \tilde{p}_+ ,
\end{aligned} \tag{80}$$

so that the Hamiltonian and spatial diffeomorphism constraints take the form

$$\begin{aligned}
\tilde{\mathcal{H}}_\tau &= -(2\pi\alpha') (\tilde{p}_\psi - T_{D1} C_{+-}^{(2)} \tilde{p}_+)^2 G^{++} + \\
&+ 2(\tilde{p}_\psi - T_{D1} C_{+-}^{(2)} \tilde{p}_+) G^{+-} (\partial_\sigma \tilde{x}^+ - T_{D1} C_{-+}^{(2)} \partial_\sigma \tilde{\psi}) - \frac{1}{2\pi\alpha'} (\partial_\sigma \tilde{x}^+ - T_{D1} C_{-+}^{(2)} \partial_\sigma \tilde{\psi})^2 G^{--} + \\
&+ \frac{1}{2\pi\alpha'} \mathbf{X} (-G_{++} (\partial_\sigma \tilde{\psi})^2 + 4\pi\alpha' G_{+-} \partial_\sigma \tilde{\psi} \tilde{p}_+ - (2\pi\alpha')^2 G_{--} (\tilde{p}_+)^2 + G_{\mu\nu} (\partial_\sigma x^\mu \partial_\sigma x^\nu) + \\
&+ (p_\mu - \frac{1}{2\pi\alpha'} (C_{\mu\sigma}^{(2)} \partial_\sigma x^\sigma - iC_{\mu+}^{(2)} \partial_\sigma \tilde{\psi} + i(2\pi\alpha') C_{\mu-}^{(2)} \tilde{p}_+)) G^{\mu\nu} \times \\
&\times (p_\nu - \frac{1}{2\pi\alpha'} (C_{\nu\rho}^{(2)} \partial_\sigma x^\rho - iC_{\nu+}^{(2)} \partial_\sigma \tilde{\psi} + i(2\pi\alpha') C_{\nu-}^{(2)} \tilde{p}_+)) , \\
\tilde{\mathcal{H}}_\sigma &= \tilde{\psi} \partial_\sigma \tilde{p}_\psi + \tilde{x}^+ \partial_\sigma \tilde{p}_+ + p_\mu \partial_\sigma x^\mu .
\end{aligned} \tag{81}$$

As the third step we perform canonical transformation along $\tilde{\psi}$ direction. We denote dual coordinate as $\tilde{\phi}$ so that we have

$$\partial_\sigma \tilde{\psi} = -(2\pi\alpha') p_{\tilde{\phi}} , \quad \tilde{p}_\psi = -\frac{1}{2\pi\alpha'} \partial_\sigma \tilde{\phi} . \tag{82}$$

Inserting these relations to $\tilde{\mathcal{H}}_\tau, \tilde{\mathcal{H}}_\sigma$ given above we derive the final form of the Hamiltonian and diffeomorphism constraints

$$\begin{aligned}
\tilde{\mathcal{H}}_\tau = & -(2\pi\alpha')\left(\frac{1}{2\pi\alpha'}\partial_\sigma\tilde{\phi} + C_{+-}^{(2)}\tilde{p}_+\right)^2 G^{++} - \\
& - \frac{2}{2\pi\alpha'}(\partial_\sigma\tilde{\phi} + (2\pi\alpha')C_{+-}^{(2)}\tilde{p}_+)G^{+-}(\partial_\sigma\tilde{x}^+ + (2\pi\alpha')C_{-+}^{(2)}p_{\tilde{\phi}}) - \frac{1}{2\pi\alpha'}(\partial_\sigma\tilde{x}^+ + (2\pi\alpha')C_{-+}^{(2)}p_{\tilde{\phi}})^2 G^{--} + \\
& + (2\pi\alpha')\mathbf{X}(-G_{++}(p_{\tilde{\phi}})^2 - 2G_{+-}p_{\tilde{\phi}}\tilde{p}_+ - G_{--}(\tilde{p}_+)^2 + \frac{1}{(2\pi\alpha')^2}G_{\mu\nu}(\partial_\sigma x^\mu\partial_\sigma x^\nu)) + \\
& + \left(p_\mu - \frac{1}{2\pi\alpha'}C_{\mu\sigma}^{(2)}\partial_\sigma x^\sigma + iC_{\mu+}^{(2)}p_{\tilde{\phi}} + iC_{\mu-}^{(2)}\tilde{p}_+\right)G^{\mu\nu}\left(p_\nu - \frac{1}{2\pi\alpha'}C_{\nu\rho}^{(2)}\partial_\sigma x^\rho + iC_{\nu+}^{(2)}p_{\tilde{\phi}} + iC_{\nu-}^{(2)}\tilde{p}_+\right), \\
\tilde{\mathcal{H}}_\sigma = & \partial_\sigma\tilde{\phi}p_{\tilde{\phi}} + \partial_\sigma\tilde{x}^+\tilde{p}_+ + p_\mu\partial_\sigma x^\mu.
\end{aligned} \tag{83}$$

Finally we fix the gauge by imposing the constraints

$$p_{\tilde{\phi}} = \frac{1}{2\pi\alpha'}, \quad x^+ = \tau. \tag{84}$$

Then we obtain that the Hamiltonian for the physical degrees of freedom should be identified as

$$\mathcal{H}_{fix} = -p_+. \tag{85}$$

From $\tilde{\mathcal{H}}_\sigma$ we obtain $\partial_\sigma\tilde{\phi} = -(2\pi\alpha')\partial_\sigma x^\mu p_\mu$. Further we see that in order to have real Hamiltonian we have to demand that $C_{\mu+}^{(2)} = C_{\mu-}^{(2)} = 0$. Then the strong constraint $\tilde{\mathcal{H}}_\tau = 0$ is equal to

$$\begin{aligned}
\mathcal{H}_\tau = & -(2\pi\alpha')(p_\mu\partial_\sigma x^\mu - C_{+-}^{(2)}\tilde{p}_+)^2 G^{++} + \\
& + 2(\partial_\sigma x^\mu p_\mu - C_{+-}^{(2)}\tilde{p}_+)G^{+-}C_{-+}^{(2)} - \frac{1}{2\pi\alpha'}G^{--}(C_{-+}^{(2)})^2 + \\
& + (2\pi\alpha')\mathbf{X}(-G_{++}\frac{1}{(2\pi\alpha')^2} - \frac{2}{2\pi\alpha'}G_{+-}\tilde{p}_+ - G_{--}(\tilde{p}_+)^2) + \mathcal{H}_x.
\end{aligned} \tag{86}$$

This equation can be solved for \tilde{p}_+ and we obtain

$$\tilde{p}_+ = \frac{\tilde{G}^{+-}}{2\pi\alpha'\tilde{G}^{++}} - \frac{2}{2\pi\alpha'\tilde{G}^{++}}\sqrt{\mathbf{K}} + p_\mu\partial_\sigma x^\mu\tilde{C}_{+-}^{(2)}, \tag{87}$$

where

$$\mathbf{K} = (\tilde{G}^{+-})^2 - \tilde{G}^{++}\tilde{G}^{--} - (2\pi\alpha')^2\tilde{G}^{++}\tilde{G}^{--}\mathbf{X}(p_\mu\partial_\sigma x^\mu)^2 - (2\pi\alpha')\tilde{G}^{++}\mathcal{H}_x, \tag{88}$$

and where we identified the new background fields

$$\begin{aligned}
\tilde{C}_{+-}^{(2)} &= -\frac{C_{+-}^{(2)}G^{++}}{G_{--}\mathbf{X} + (C_{+-}^{(2)})^2G^{++}} , \\
\tilde{G}^{++} &= -\mathbf{X}G_{--} - (C_{+-}^{(2)})^2G^{++} , \\
\tilde{G}^{+-} &= G_{+-}\mathbf{X} - (C_{+-}^{(2)})^2G^{+-} , \\
\tilde{G}^{--} &= -G^{--}(C_{+-}^{(2)})^2 - \mathbf{X}G_{++} .
\end{aligned} \tag{89}$$

As in section (3) we compare this result with the uniform gauge fixed D1-brane when we consider canonical dual Hamiltonian constraint

$$\begin{aligned}
\mathcal{H}_\tau &= (2\pi\alpha')\Pi_+G^{++}\Pi_+ + 2(2\pi\alpha')\Pi_+G^{+-}\Pi_- + (2\pi\alpha')\Pi_-G^{--}\Pi_- + \\
&+ \frac{1}{2\pi\alpha'}\mathbf{X} \left(G_{++}(\partial_\sigma x^+)^2 - 4\pi\alpha'G_{+-}\partial_\sigma x^+p_\psi + (2\pi\alpha')^2G_{--}(p_\psi)^2 \right) + \mathcal{H}_x , \\
\mathcal{H}_\sigma &= p_+\partial_\sigma x^+ + p_\psi\partial_\sigma\psi + p_\mu\partial_\sigma x^\mu ,
\end{aligned} \tag{90}$$

where

$$\Pi_+ = p_+ + C_{+-}^{(2)}p_\psi , \quad \Pi_- = -\frac{1}{2\pi\alpha'}(\partial_\sigma\psi + C_{+-}^{(2)}\partial_\sigma x^+) , \quad \Pi_\mu = p_\mu - \frac{1}{2\pi\alpha'}C_{\mu\nu}^{(2)}\partial_\sigma x^\nu . \tag{91}$$

Then we perform the gauge fixing

$$x^+ = \tau , \psi = \sigma \tag{92}$$

and we obtain

$$\Pi_- = -\frac{1}{2\pi\alpha'} , \quad p_\psi = -p_\mu\partial_\sigma x^\mu \tag{93}$$

and hence the Hamiltonian constraint is equal to

$$\begin{aligned}
0 &= \mathcal{H}_\tau = (2\pi\alpha')\Pi_+G^{++}\Pi_+ - 2\Pi_+G^{+-} + \frac{1}{2\pi\alpha'}G^{--} + \\
&+ (2\pi\alpha')\mathbf{X}G_{--}(p_\mu\partial_\sigma x^\mu)^2 + \mathcal{H}_x
\end{aligned} \tag{94}$$

that can be solved for p_+ as

$$\begin{aligned}
p_+ &= \frac{G^{+-}}{2\pi\alpha'G^{++}} - C_{+-}^{(2)}(p_\mu\partial_\sigma x^\mu) \\
&- \frac{2}{2\pi\alpha'G^{++}}\sqrt{((G^{+-})^2 - G^{++}G^{--}) - (2\pi\alpha')^2G^{++}G_{--}\mathbf{X}(p_\mu\partial_\sigma x^\mu)^2 - (2\pi\alpha')G^{++}\mathcal{H}_x} .
\end{aligned} \tag{95}$$

Now comparing (95) with (87) we see that these two gauge fixed Hamiltonians have the same form when we perform the identification of the background fields as was given in (89). It is important to stress that the transformation rules (89) coincide with the rules that were derived in [11] when the double Wick rotation was performed on the world-volume of uniform gauge fixed D1-brane action. We also see that \mathbf{X} does not transform again with agreement with [11]. Finally using the arguments given in section (4) we can argue that (89) are in agreement with [9] when $C_{+-}^{(2)} = 0$. Then we can rewrite (89) into the form

$$\begin{aligned}\frac{1}{\sqrt{(C^{(0)})^2 + e^{-2\Phi}}} \tilde{G}^{++} &= -\sqrt{e^{-2\Phi} + (C^{(0)})^2} G_{--} , \\ \frac{1}{\sqrt{(C^{(0)})^2 + e^{-2\Phi}}} \tilde{G}^{+-} &= G_{+-} \sqrt{(C^{(0)})^2 + e^{-2\Phi}} , \\ \frac{1}{\sqrt{(C^{(0)})^2 + e^{-2\Phi}}} \tilde{G}^{--} &= \sqrt{(C^{(0)})^2 + e^{-2\Phi}} G_{++} .\end{aligned}\tag{96}$$

We immediately see that these metric components correspond to the S-dual metric when we used the equivalence between D1-brane action and the fundamental string action in S-dual background. Hence (96) precisely correspond to the rules derived in [9] when are applied for the string moving in S-dual background.

In summary, we have shown that there is a equivalence between sequence of canonical transformations and target space Wick rotation on one side and the double Wick rotation on the gauge fixed D1-brane action on the another side. On the other hand we also see that the components of the two form field $C_{\mu\nu}^{(2)}$ that are transverse to the directions where the canonical transformations were performed do not transform while their change the sign in case of double Wick rotation of the uniform gauge fixed D1-brane action. On the other hand we can argue as in section (2) that the transformation $C^2 \rightarrow -C^{(2)}$ maps one solutions of the supergravity equations of motion to another one (together with $B \rightarrow -B$) and hence we see that there is an equivalence between double Wick rotation on the world-volume of uniform gauge fixed D1-brane and sequence of transformations: "canonical transformation-target space double Wick rotation-canonical transformation- $C^{(2)} \rightarrow -C^{(2)}, B \rightarrow -B$).

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